

Abrupt Changes in the Dynamics of Quantum Disentanglement

F. Lastra¹, G. Romero¹, C. E. López¹, M. França Santos², and J.C. Retamal¹

¹*Departamento de Física, Universidad de Santiago de Chile, USACH, Casilla 307 Correo 2, Santiago, Chile.*

²*Departamento de Física, Universidade Federal de Minas Gerais, Belo Horizonte, Brazil.*

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Entanglement evolution in high dimensional bipartite systems under dissipation is studied. Discontinuities for the time derivative of the lower bound of entanglement of formation is found depending on the initial conditions for entangled states. This abrupt changes along the evolution appears as precursors of entanglement sudden death.

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Entanglement is a cornerstone of modern quantum physics [1, 2]. The evolution of entanglement in open quantum systems is a matter of increasing interest and new phenomena have been predicted [3, 4, 5, 6, 7, 8, 9, 10]. One of the most outstanding effects arises when entanglement vanishes long before coherence is lost. It has been pointed out that systems composed of two qubits in a noisy environment can lose its entanglement in finite time, a phenomena named as Entanglement Sudden Death (ESD), even though full decoherence happens asymptotically. This feature appears for certain classes of states of two qubits under the action of independent reservoirs. Examples of these classes are the so-called “X” mixed states as well as some particular types of non-maximally entangled pure states [7].

The purpose of this work is to explore dynamical behavior of entangled states in larger bipartite systems under the action of independent reservoirs. We show that unlike the case of two qubits, $3 \otimes 3$ systems may present not only ESD, but also intermediate abrupt changes in the disentanglement dynamics, i.e. the rate in which a given state loses its entanglement may change through-

out the dissipative process even though coherence is lost in a constant rate. We show that these rate changes are associated with sudden changes in the rank of the partially transposed density matrix, which also provides an explanation for the sudden death of entanglement.

We analyze the disentanglement dynamics for different initial states and show that abrupt changes may be present or not depending on the variation of a small number of parameters. We also recover the result for two qubits when preparing the initial state in a $2 \otimes 2$ subspace of the whole system. Finally, we interpret these results in terms of changes in the set of entanglement witnesses appropriated for the characterization of the entangled state in each part of the dynamics.

In this work we use a general measurement for the lower bound of Entanglement of Formation (EOF) for a mixed state in $m \otimes n$ dimensions, which has been recently proposed [11]. This proposal is based on the comparison between two major criteria: (i) the positivity under partial transposition (PPT criterion) [12, 13] and (ii) the realignment criterion [14, 15]. EOF for $m \otimes n$ -dimensional systems ($m \leq n$) is defined as [11]

$$E(\rho) \geq \begin{cases} 0 & \text{if } \Lambda = 1, \\ H_2[\gamma(\Lambda)] + [1 - \gamma(\Lambda)] \log_2(m-1) & \text{if } \Lambda \in \left[1, \frac{4(m-1)}{m}\right], \\ \frac{\log_2(m-1)}{m-2}(\Lambda - m) + \log_2(m) & \text{if } \Lambda \in \left[\frac{4(m-1)}{m}, m\right], \end{cases} \quad (1)$$

where m is the dimension of the first subsystem and γ is given by

$$\gamma(\Lambda) = \frac{1}{m^2} \left[\sqrt{\Lambda} + \sqrt{(m-1)(m-\Lambda)} \right]^2, \quad (2)$$

with $\Lambda = \max(\|\rho^{TA}\|, \|R(\rho)\|)$ and $H_2(x) = -x \log(x) - (1-x) \log(1-x)$. The trace norm $\|\cdot\|$ is defined by $\|G\| = \text{tr}(GG^\dagger)^{\frac{1}{2}}$. The matrix ρ^{TA} is the partial transpose with respect to the subsystem A , that is, $\rho_{ik,jl}^{TA} = \rho_{jk,il}$, and the matrix $R(\rho)$ is defined as $R(\rho)_{ij,kl} = \rho_{ik,jl}$.

The PPT criterion says that $\rho^{TA} \geq 0$ for a separable state [12]. On the other hand, the realignment criterion

says that a realigned version of ρ , for a separable state must satisfy the condition: $\|R(\rho)\| \leq 1$. These conditions state that entanglement exists for $\Lambda > 1$. The maximum values that $\Lambda(t)$ can assume depend on the dimensions of the bipartite systems. For example, for a maximal two qutrits entangled state $\Lambda = 3$, for two qubits $\Lambda = 2$. The minimum value for a separable state is always $\Lambda = 1$. In this work we use this quantity to study the time evolution of entanglement in the presence of dissipation.

Let us consider entangled quantum states of two qutrits, with at most two excitations, in the presence of dissipation at zero temperature. Such situation can

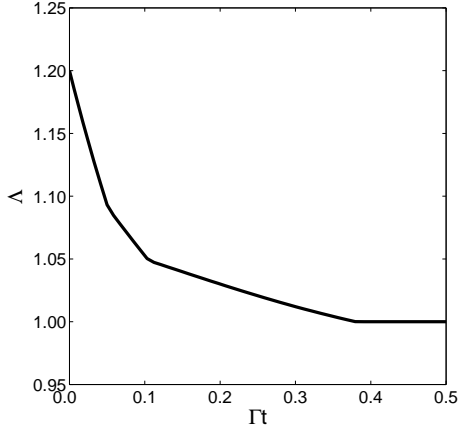


FIG. 1: Evolution of $\Lambda(t)$ for the initial mixed state of Eq.(4) with $\lambda = 0.1$ as a function of the adimensional time Γt .

be conveniently described by the evolution equation:

$$\dot{\rho} = \sum_{i=1,2} \frac{\Gamma_i}{2} \left[2c_i \rho c_i^\dagger - c_i^\dagger c_i \rho - \rho c_i^\dagger c_i \right], \quad (3)$$

where c_i, c_i^\dagger describes annihilation and creation operators for bosonic modes and ρ is a $3 \otimes 3$ density matrix in the computational basis $\{|0\rangle, |1\rangle, |2\rangle\} \otimes \{|0\rangle, |1\rangle, |2\rangle\}$ of both qutrits.

Let us consider at first glance a class of initially mixed states of two qutrits which corresponds to a modification of a maximally entangled state given as follows:

$$\rho(0) = \frac{1}{3} \begin{pmatrix} 1 & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & \lambda \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \lambda \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \lambda & 0 & 0 & 0 & \lambda & 0 & 0 & 0 & 1 \end{pmatrix}, \quad (4)$$

where λ is a real parameter ranging from $0 < \lambda < 1$. In the extreme cases, $\lambda = 0$, we have a separable state whereas for $\lambda = 1$ we have a maximally entangled state. The Eq.(3) can be solved for arbitrary decay constants, but for simplicity we reduce the problem to the simplest case $\Gamma_1 = \Gamma_2 = \Gamma$. By a numerical calculation we realize that $\|\rho^{TA}\| \geq \|R(\rho)\|$ for all times, so that, we need to concentrate only in $\Lambda(t) = \|\rho^{TA}\|$.

Fig. 1 shows the evolution of $\Lambda(t)$ for $\lambda = 0.1$. As we observe, $\Lambda(t)$ undergoes sudden changes along its evolution exhibiting discontinuous derivatives, finally evolving to a situation where entanglement abruptly dies. As compared with the case of two qubits a richer dynamical behavior of entanglement appears. From the definition of $\Lambda(t)$, this feature must be closely connected with the temporal dependence of the eigenvalues of $M = \rho^{TA} \cdot (\rho^{TA})^\dagger$. In our case both analytical and numerical calculations

of the eigenvalues of the matrix M can be carried out. From numerical calculations we realize that the abrupt changes of entanglement evolution are dominated by the behavior of a restricted number of eigenvalues given by:

$$\begin{aligned} E_1(t) &= (\rho_{12,12})^2 + (\rho_{11,22})^2 - 2\rho_{12,12}\rho_{11,22}, \\ E_2(t) &= (\rho_{00,11})^2 + (\rho_{01,01})^2 - 2\rho_{00,11}\rho_{01,01}, \\ E_3(t) &= (\rho_{00,22})^2 + (\rho_{02,02})^2 - 2\rho_{00,22}\rho_{02,02}, \end{aligned} \quad (5)$$

where

$$\begin{aligned} \rho_{12,12} &= \frac{2}{3} (e^{-3\Gamma t} - e^{-4\Gamma t}) \\ \rho_{11,22} &= \frac{\lambda}{3} e^{-3\Gamma t} \\ \rho_{00,11} &= \frac{2\lambda}{3} e^{-3\Gamma t} - \frac{4\lambda}{3} e^{-2\Gamma t} + \lambda e^{-\Gamma t} \\ \rho_{01,01} &= 2e^{-3\Gamma t} - \frac{7}{3} e^{-2\Gamma t} - \frac{2}{3} e^{-4\Gamma t} + e^{-\Gamma t} \\ \rho_{00,22} &= \frac{\lambda}{3} e^{-2\Gamma t} \\ \rho_{02,02} &= -\frac{2}{3} e^{-3\Gamma t} + \frac{1}{3} e^{-4\Gamma t} + \frac{1}{3} e^{-2\Gamma t} \end{aligned} \quad (6)$$

are the density matrix elements $\rho_{ij,kl}$. These eigenvalues (5) are plotted in Fig. 2, where we observe that the times where they vanish are in exact agreement with the times where abrupt changes in the entanglement evolution appear. From Eqs. (5), these times can be analytically calculated in terms of the parameter λ :

$$t_1 = \ln(2/(2-\lambda)), t_2 = \ln(1/(1-\lambda)), t_3 = \ln(1/(1-\sqrt{\lambda})). \quad (7)$$

Fig. 3 shows the smooth behaviors of these times as a function of the parameter λ defining particular two qutrits mixed states. From this picture we realize that the abrupt changes in the dynamics of entanglement will appear for all values of λ in the interval $[0, 1]$. In particular for the maximally entangled state with $\lambda = 1$, there is sudden change for $t_1 = \ln 2$, and the time of the second and third sudden change, which is the ESD, goes to infinite, showing that the entanglement decays asymptotically. Note that this result differs substantially from its two-qubit counterpart where the corresponding maximally entangled states disentangle smoothly [7]. Also note that these abrupt changes can be mathematically interpreted as discontinuities of the derivative for the expression $\Lambda(t) = \sum_{i=1}^9 \sqrt{E_i(t)}$, and a sudden change in the evolution of Λ occurs whenever one of the nine eigenvalues E_i becomes zero, as observed in Fig. 2

The analysis to explain these particular sudden changes in the evolution of entanglement has been done in terms of the eigenvalues of the matrix M . However, we can also understand these abrupt changes in $\Lambda(t)$ by observing the behavior of the eigenvalues of the partial transpose matrix ρ^{TA} . In our case only three eigenvalues give us information about these sudden changes and are plotted in Fig. 4. We notice that these eigenvalues change from negative to positive values for specific times which are in agreement with the sudden changes in the

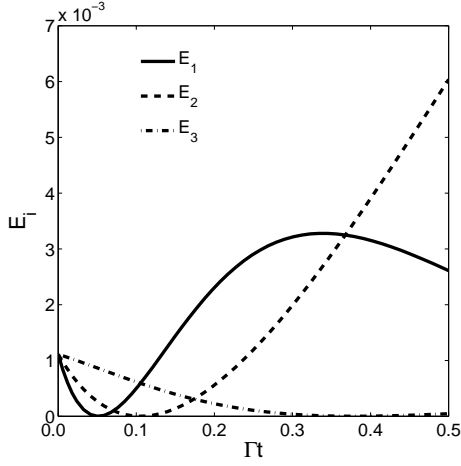


FIG. 2: Relevant eigenvalues of matrix M for the case shown in Fig. 1.

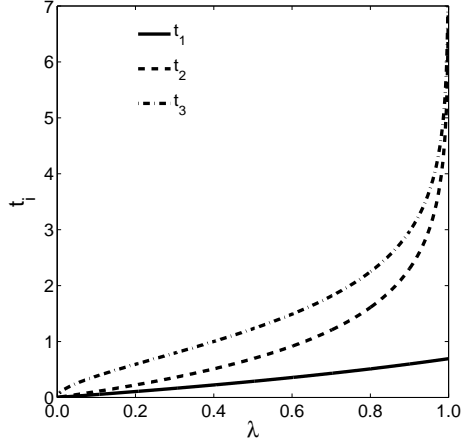


FIG. 3: Times for sudden changes in the dynamics of entanglement as a function of the parameter λ . Solid line corresponds to the first sudden change, dashed line corresponds to the second sudden change, and dot-dashed line corresponds to the sudden death.

entanglement evolution. In other words, the disentanglement rate changes whenever the rank of the partially transposed matrix changes abruptly. We can also associate to each eigenvalue of ρ^{TA} a corresponding entanglement witness operator such that $\alpha_i(t) = \text{Tr}(W_i \rho(t))$ with $i = 1, 2, 3$ and each W_i is given by

$$\begin{aligned} W_1 &= \frac{1}{2} [|21\rangle\langle 21| - |11\rangle\langle 22| - |22\rangle\langle 11| + |12\rangle\langle 12|], \\ W_2 &= \frac{1}{2} [|10\rangle\langle 10| - |00\rangle\langle 11| - |11\rangle\langle 00| + |01\rangle\langle 01|], \\ W_3 &= \frac{1}{2} [|02\rangle\langle 02| - |00\rangle\langle 22| - |22\rangle\langle 00| + |20\rangle\langle 20|]. \end{aligned} \quad (8)$$

At $t = 0$, all three operators can be used to identify entanglement in ρ . As time goes by, they consecutively lose this capacity until there is no entanglement left. This suggests a geometrical interpretation to the phenomena here described which will be explored in further publications.

It is interesting to compare the case analyzed previ-

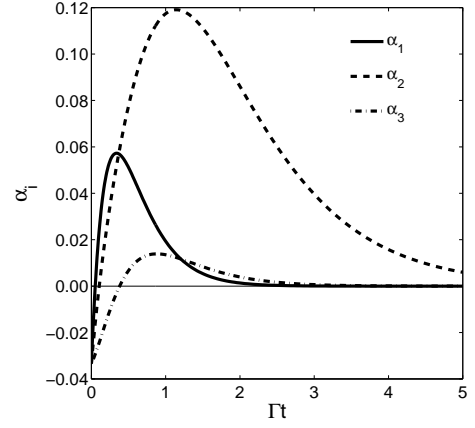


FIG. 4: Relevant eigenvalues of partial transpose matrix ρ^{TA} .

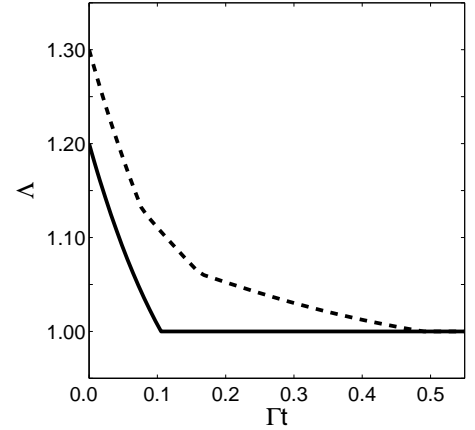


FIG. 5: Entanglement evolution for the case shown in Fig (1) (dashed line) for $\lambda = 0.15$ and for a initial state in the subspace $\{|11\rangle, |22\rangle\}$ (solid line) for $\chi = 0.2$

ously with that of an initial state restricted to a two-dimensional subspace $\rho = (1/2)(|11\rangle\langle 11| + |22\rangle\langle 22| + \chi|11\rangle\langle 22| + \chi|22\rangle\langle 11|)$. Fig. 5 shows the evolution of the entanglement for $\chi = 0.2$ as compared with the state in Eq. 4 for $\lambda = 0.15$. We observe that in the case of the initial condition restricted to two dimensional subspace we have only one abrupt change in the evolution corresponding to the ESD which resembles the behavior observed for two qubits [7]. If we look at the eigenvalues of M we see that at the time when Λ goes to zero there is also an eigenvalue that goes to zero, indicating that an abrupt change occurs. A similar conclusion could actually be obtained when looking at the negativity instead of Concurrence for the case of two qubits.

In addition we could explore the entanglement evolution for initial non maximally pure entangled states, for example, $|\Phi\rangle = \alpha|00\rangle + \beta|11\rangle + \gamma|22\rangle$. For the sake of simplicity we consider α, β and γ real positive numbers. In this case, a richer dynamics for the entanglement can be observed. Depending on the choice of the amplitudes we can have asymptotic decay, sudden death, sudden changes or a combination of them. Times corre-

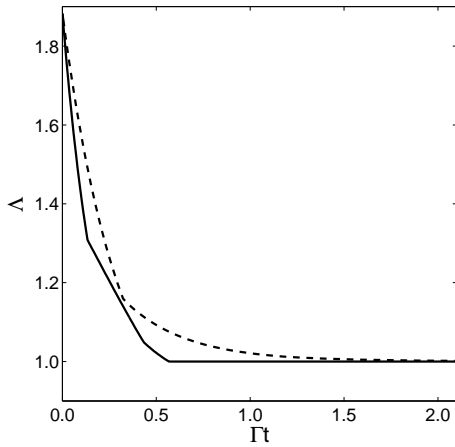


FIG. 6: Entanglement evolution for the non maximal pure state. Dashed line corresponds to $\alpha = 0.2386$, $\beta = 0.9545$, $\gamma = 0.1790$ and solid line corresponds to $\alpha = 0.1790$, $\beta = 0.2386$, $\gamma = 0.9545$.

sponding to the sudden changes and the ESD time are given by:

$$\begin{aligned} t_1 &= -\frac{1}{\Gamma} \ln \left(1 - \frac{\beta}{2\gamma} \right), \quad t_3 = -\frac{1}{\Gamma} \ln \left(1 - \frac{\alpha}{\gamma} \right), \\ t_2 &= -\frac{1}{\Gamma} \ln \left[1 - \frac{\beta}{3\gamma} \left(1 + \frac{1}{(2Z)^{1/3}} - \left(\frac{Z}{2} \right)^{1/3} \right) \right], \end{aligned} \quad (9)$$

where

$$Z = 5 - 27 \frac{\alpha\gamma}{\beta^2} + 3\sqrt{3} \left(1 - 10 \frac{\alpha\gamma}{\beta^2} + 27 \left(\frac{\alpha\gamma}{\beta^2} \right)^2 \right)^{1/2}. \quad (10)$$

From these expressions we realize that the entanglement dynamics exhibit: (a) asymptotic decay for ($\alpha \geq \beta > \gamma$), (b) one sudden change and asymptotic decay for ($\beta \geq \alpha \geq \gamma$, or $\alpha > \gamma > \beta$), (c) two sudden changes and asymptotic decay for ($\beta > \gamma > \alpha$), and (d) two sudden changes and ESD for ($\gamma > \beta > \alpha$). Fig. 6 shows two particular dynamics evolution for the cases (b) and (d).

In summary we have studied the evolution of entanglement for high dimensional dissipative quantum systems. By evaluating the entanglement contained in the system using the Chen, Albeverio and Fei measure we have observed outstanding new effects. Quantum correlations undergo abrupt changes as precursors of ESD. These can be characterized by observing the eigenvalues of the M matrix which defines the amount of entanglement for the quantum system. The dynamical changes are related to sudden changes in the rank of the Matrix M . Similar behavior can be found for both initially mixed or pure states and the ESD is recovered as a particular case of these sudden dynamical changes.

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